Closing *Tues*: 10.2
Closing Fri: 3.5(1)(2)

3.5 Implicit Differentiation (*continued*)

Given any equation of the form:

$$F(x,y) \ = \ 0\,,$$

we think of *y* as an *implicit* function of *x*

$$F(x,y(x)) \ = \ 0$$

and differentiate directly (**correctly using the chain rule as we go**) to find dy/dx.

Entry Task: Find the equation for the tangent line to

$$y^2 = x$$

at $(x,y) \ = \ (4,-2)$.

OPTION 1

$y^2 = x$

$\hookrightarrow y = \pm\sqrt{x}$

SINCE $y<0$, USE

$y = -\sqrt{x} = -x^{1/2}$

$y' = -\frac{1}{2}x^{-1/2} = -\frac{1}{2\sqrt{x}}$

$y'(4) = -\frac{1}{2\sqrt{4}}$

$= -\frac{1}{4}$

OPTION 2

$\frac{d}{dx}\left[y^2 = x\right]$

$2y\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{2y}$

$\frac{1}{2(-2)} = -1/4$

$y = -\frac{1}{4}(x-4) + -2$

Find dy/dx.

1. $x^4y + y^3 = x$

$y = f(x)$

$f'(x) = \frac{dy}{dx} = y'$

$\frac{d}{dx}\left[x^4 \underbrace{f(x)}_{y} + (\underbrace{f(x)}_{y})^3 = x\right]$

$4x^3 y + x^4 \frac{dy}{dx} + 3(y)^2 \cdot \frac{dy}{dx} = 1$

$(x^4 + 3y^2)\frac{dy}{dx} = 1 - 4x^3 y$

$\boxed{\frac{dy}{dx} = \frac{1-4x^3y}{x^4+3y^2}}$

2. $xe^y + \tan(x) + \sin(y) = 1$

$e^y + xe^y \frac{dy}{dx} + \sec^2(x) + \cos(y)\frac{dy}{dx} = 0$

$(xe^y + \cos(y))\frac{dy}{dx} = -e^y - \sec^2(x)$

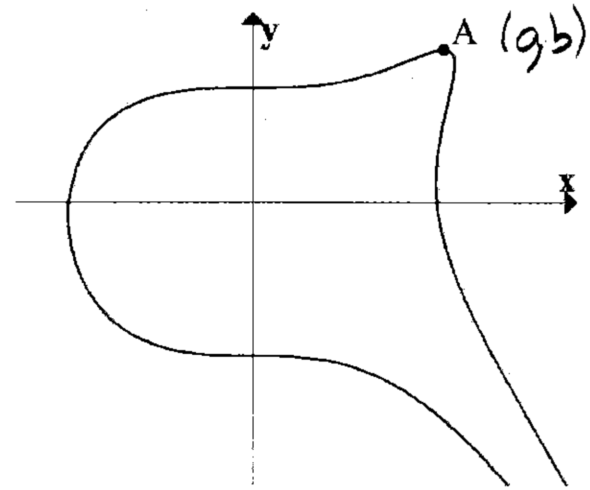
$\boxed{\frac{dy}{dx} = -\frac{(e^y + \sec^2(x))}{(xe^y + \cos(y))}}$

Old Midterm Question:

Consider the curve implicitly defined by

$$\frac{d}{dx}\left[\ (x^3 - y^2)^2 + e^y = 4.\right]$$

Find the (x, y) coordinates of the point A shown (highest point on the curve).



WANT TO KNOW WHERE $dy/dx \overset{?}{=} 0$

$$\boxed{2(x^3-y^2)\left(3x^2-2y\frac{dy}{dx}\right)+e^y\frac{dy}{dx}=0}$$

a b a 0 0

WANT $2(a^3-b^2)\,3a^2 \overset{?}{=} 0 \Rightarrow a=0$ NO or $\boxed{a^3-b^2=0}$

ALSO $(a^3-b^2)^2+e^b=4$

$\Rightarrow$ $e^b=4$ $\Rightarrow$ $\boxed{b=\ln(4)}$

$a^3=b^2$
$a=b^{2/3}$

$$\boxed{a=(\ln(4))^{2/3}}$$

Inverse Functions: We write inverse functions as $y = f^{-1}(x)$ which is equivalent to $f(y) = x$.

We can implicitly differentiate

$$\frac{d}{dx}[f(y) = x] \Rightarrow f'(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)}$$

Examples: Find dy/dx

1. $y = \sqrt{x}$ $\iff$ $y^2 = x$ $(x \geq 0)$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x}}$$

$2.\ y = \sin^{-1}(x)$ $\quad -\pi/2 \le x \le \pi/2$

$\sin(y) = x$

$\cos(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cos(y)}$

$\sin^2(y) + \cos^2(y) = 1$

$\Rightarrow \cos^2(y) = 1 - \sin^2(y)$

$\Rightarrow \cos(y) = \pm\sqrt{1-\sin^2(y)}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sin(y))^2}}$

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

$3.\ y = \tan^{-1}(x)$ $\quad -\pi/2 < x < \pi/2$

$\tan(y) = x$

$\sec^2(y)\frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$\tan^2(y) + 1 = \sec^2(y)$

$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$	$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{\mathrm{x}\sqrt{x^2-1}}$	$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{\mathrm{x}\sqrt{x^2-1}}$

- *Note*: The formulas all assume the principal domains as defined in our textbook.

Now you can use these *shortcuts*.

Exercise: Find dy/dx

$$y = \tan^{-1}(e^{3x})$$

$$\frac{dy}{dx} = \frac{1}{1+(e^{3x})^2} \cdot e^{3x} \cdot 3 = \frac{3e^{3x}}{1+e^{6x}}$$